

Approximate Homogenization of Fully Nonlinear Elliptic PDEs: Estimates and Numerical Results for Pucci Type Equations

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Abstract We are interested in the shape of the homogenized operator $\bar{F}(Q)$ for PDEs which have the structure of a nonlinear Pucci operator. A typical operator is $H^{a_1, a_2}(Q, x) = a_1(x)\lambda_{\min}(Q) + a_2(x)\lambda_{\max}(Q)$. Linearization of the operator leads to a non-divergence form homogenization problem, which can be solved by averaging against the invariant measure. We estimate the error obtained by linearization based on semi-concavity estimates on the nonlinear operator. These estimates show that away from high curvature regions, the linearization can be accurate. Numerical results show that for many values of Q , the linearization is highly accurate, and that even near corners, the error can be small (a few percent) even for relatively wide ranges of the coefficients.

Keywords Viscosity solutions · Partial differential equations · Homogenization · Pucci maximal operator

1 Introduction

In this article we consider fully nonlinear, uniformly elliptic PDEs $F(Q, x)$. We are interested in approximating the homogenized operator $\bar{F}(Q)$. We focus on Pucci-type PDE operators in two dimensions. The restriction to two dimensions is for computational simplicity and also for visualization purposes. We consider periodic coefficients, although in our numerical experiments we obtained very similar results with random coefficients.

The approach we take is to linearize the operator about the value Q , and to homogenize the linearized operator $\bar{L}(Q)$. The solution of the linear homogenization problem can be expressed (and in some cases solved analytically) by averaging against the invariant measure. The result is given by

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$$\overline{L^Q}(Q) = \int F(Q, x)\rho^Q(x)dx$$

where ρ^Q is the invariant measure of the corresponding linear problem. We estimate the linearization error

$$E(Q) \equiv \overline{F}(Q) - \overline{L^Q}(Q),$$

For convex operators, the analysis gives a one sided bound on the error. In general, we obtain upper or lower bounds on the error, which depend on generalized semi-concavity/convexity estimates of F , as well as on the solution of the cell problem u^Q for the nonlinear problem. These results are stated in Theorem 2.4 below.

For theoretical results on nonlinear homogenization, we refer to the review [5] as well as recent works on rates of convergence (for example [1]). There are fewer works which aim to determine the values $\overline{F}(Q)$. Few analytical results are available. Numerical homogenizing results for Pucci type operators can be found in [4] using a least-squares formulation. We also mention numerical work by [11, 14, 16] in the first order case, as well as [8] in the second order linear non-divergence case.

The typical operator we consider herein is defined next. Below, we consider more operators, including the usual convex Pucci Maximal operator.

Definition 1.1 (Fully nonlinear elliptic operator $F(Q, x)$ and linearization) We are given $F : \mathcal{S}^d \times \mathbb{T}^d \rightarrow \mathbb{R}$ which is uniformly elliptic, Lipschitz continuous in the first variable and bounded in the second variable. Suppose for a given Q , that $\nabla_Q F(Q, x)$ exists for all x . Write

$$L^Q(M, x) = \nabla_Q F(Q, x) \cdot (M - Q) + F(Q, x) \tag{1}$$

for the affine approximation to F at Q .

Given $Q \in \mathcal{S}^d$, write, for $d = 2$, $\lambda_{\min}(Q)$ and $\lambda_{\max}(Q)$ for the smaller, and larger eigenvalues of Q , respectively.

Example 1.2 (Typical PDE operator) Given $\delta > 0$, and periodic functions $a_1(y), a_2(y) \geq \delta$. Define the homogeneous order one PDE operator

$$H^{a_1, a_2}(Q, x) = a_1(x)\lambda_{\min}(Q) + a_2(x)\lambda_{\max}(Q) \tag{2}$$

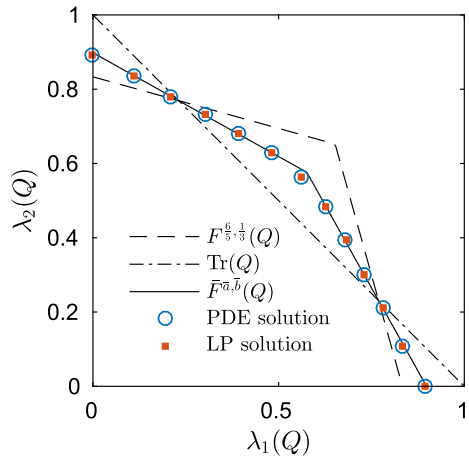
Suppose Q has unit eigenvectors v_1, v_2 corresponding to the eigenvalues $\lambda_{\min}(Q), \lambda_{\max}(Q)$, respectively. Then the linearization at Q , of H^{a_1, a_2} is given by

$$L^Q(M, x) = a_1(x)v_1^T M v_1 + a_2(x)v_2^T M v_2 \tag{3}$$

Remark 1.3 (Typical results) We consider the case of coefficients which are either (i) periodic checkerboards or (ii) random checkerboards. We compute both the nonlinear homogenization $\overline{F}(Q)$ and the homogenized linear operator $\overline{L^Q}(Q)$. In practice, the numerically computed error is insignificant, less than $1e-8$ for values of Q , away from regions of high curvature of F with respect to Q . Areas where the error is significant correspond to regions where the semi-concavity constants are large. A typical result is displayed in Fig. 1. The solid line is a level set of the homogenized linear operator $\overline{L^Q}(Q)$. The dots are numerical computations of $\overline{F}(Q)$. The error is very small, except at one point, which corresponds to a corner of the operator. (Dashed lines indicate underlying operators which comprise $F(Q, x)$).

Our analysis depends on the shape of $F(Q, x)$ in Q , but not on the pattern of the coefficients in x . We also considered the case of stripe coefficients. For separable examples, the linear

Fig. 1 Plot of a single level set of $L^Q(Q)$ and $\bar{F}(Q)$. This example is typical. In this case the coefficients are on a random checkerboard. The error is only visible near the corner of the level set of the operator, $F(Q, y)$, a Pucci-type operator (see Definition 3.1 below). The details of the coefficients can be found in Sect. 4



approximation is still effective. However, we also found nonseparable examples where the linear approximation is poor, which we will address in a companion paper [10] with a closer bound.

1.1 Background: Cell Problem and Linear Homogenization

In this section, we review background material on the cell problem for the nonlinear PDE, and on linear homogenization.

Given $a(y) : \mathbb{T} \rightarrow \mathbb{R}$, positive, $a(y) > 0$, write $\text{HM}(a) = \left(\int_{\mathbb{T}} \frac{dy}{a(y)} \right)^{-1}$ for the harmonic mean of a .

In the linear case, \bar{L} can be found by averaging against the invariant measure, by solving the adjoint equation (see [2] or [8]), which yields the following formula.

Lemma 1.4 (Linear Homogenization Formula) *The separable linear operator $L(M, x) = a(x)A_0 : M + f(x)$ has invariant measure $\rho(x) = \text{HM}(a)/a(x)$ and homogenizes to $\bar{L}(Q) = \text{HM}(a)A_0 : Q + \bar{f}$, where $\bar{f}(x) = \int f(x) d\rho(x)$.*

For the nonlinear operator F , the homogenized operator is given by solving the cell problem, see [7].

Definition 1.5 (Solution of the cell problem) *Given F uniformly elliptic, for each $Q \in S^d$, there is a unique value $\bar{F}(Q)$ and a periodic function $u^Q(y)$ which is a viscosity solution of the cell problem*

$$F(Q + D^2u^Q(y), y) = \bar{F}(Q). \tag{4}$$

Lemma 1.6 (Homogenization of linearized operator) *Consider the nonlinear elliptic operator $F(Q, x)$, and suppose for a given Q , that $\nabla_Q F(Q, x)$ exists for all x . The corresponding linearization at Q is given by (1). Let ρ^Q be the corresponding unique invariant probability measure, which is the solution of the adjoint equation*

$$D^2 : (\nabla_Q F(Q, y)\rho^Q(y)) = 0, \tag{5}$$

interpreted in the weak sense. Then $\overline{L^Q}(Q)$, the homogenized linearized operator evaluated at Q , is given by

$$\overline{L^Q}(Q) = \int_{\mathbb{T}^d} F(Q, y) \, d\rho^Q(y). \tag{6}$$

Proof The invariant measure ρ^Q solves (5), see [2] or [8]. Apply (1) at $M = Q$ and then integrate against ρ^Q to obtain the result. \square

2 Main Result

2.1 Generalized Semiconcavity Estimates on the Operators

Consider the uniformly elliptic operator $F(Q, x)$, where $Q \in \mathcal{S}^d$ and $x \in \mathbb{T}^d$. We assume the following.

Assumption 2.1 (*Quadratically dominated for $F(Q, x)$*) Let F be as in Definition 1.1. Suppose for a given Q , that $\nabla_Q F(Q, x)$ exists for all x . Write $\|Q\|$ for the Frobenious norm of Q . We say that F is *quadratically dominated above* at Q if there is a bounded function $C^+(Q, x) : \mathbb{T}^d \rightarrow \mathbb{R}$ such that

$$F(M, x) - L^Q(M, x) \leq C^+(Q, x) \frac{\|M - Q\|^2}{2}, \quad \text{for all } (M, x) \in \mathcal{S}^d \times \mathbb{T}^d \tag{7}$$

and similarly, F is *quadratically dominated below* at Q if there is a bounded function $C^-(Q, x) : \mathbb{T}^d \rightarrow \mathbb{R}$ such that

$$F(M, x) - L^Q(M, x) \geq C^-(Q, x) \frac{\|M - Q\|^2}{2}, \quad \text{for all } (M, x) \in \mathcal{S}^d \times \mathbb{T}^d \tag{8}$$

Remark 2.2 If F is convex in Q , then $C^-(Q, y) = 0$. Similarly if F is concave in Q , $C^+(Q, y) = 0$. More generally if F is semi-concave, or semi-convex in Q , then we can set $C^\pm(Q, y) = C^\pm(y)$, to be a constant independent of Q . However, we require the definition above for when the semi-concavity or semi-convexity conditions in Q do not hold, as is the case for the Pucci-type operators defined below.

Example 2.3 Let $x \in \mathbb{R}$ and set $f(x) = \max\{ax, bx\}$. Since f is convex, we can take $C^-(x) = 0$ in (8). We claim that for $x \neq 0$, (7) holds with

$$C^+(x) = \frac{|a - b|}{2x}, \tag{9}$$

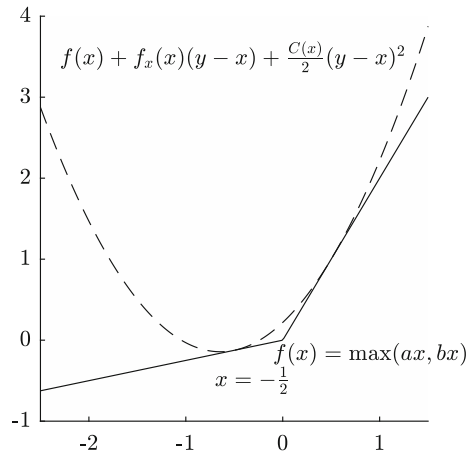
and this is the best constant. See Fig. 2.

Derivation of (9) Expand $f(x + y)$ about the point x , for $x \neq 0$. To test (7), replace the inequality with an equality to obtain a quadratic equation. By requiring that there is only one root, we obtain an equation for the discriminant of the quadratic, which can be solved to obtain the result. \square

2.2 Main Theorem

Theorem 2.4 *Suppose F satisfies Assumption 2.1 and $u^Q \in C^{2,\alpha}(\mathbb{T}^d)$ is a classical solution. Let $\overline{F}(Q)$ be the homogenized operator at Q and let u^Q be the corresponding solution of*

Fig. 2 For the simple example $f(x) = \max\{ax, bx\}$, the semi-concavity constant is $C(x) = C^+(x) = \frac{|a-b|}{2x}$



the cell problem given by (4). Let the homogenization of the linearization of the operator be given by (6) and let $\rho^Q(y)$ be the corresponding invariant measure of the linearized problem (1). Write

$$\overline{C^\pm}(Q) = \frac{1}{2} \int C^\pm(Q, y) \|D^2u^Q(y)\|^2 d\rho^Q(y)$$

Then

$$\overline{C^-}(Q) \leq \overline{F}(Q) - \overline{L^Q}(Q) \leq \overline{C^+}(Q) \tag{10}$$

Remark 2.5 In the examples we consider below, $C^\pm(Q, y) \rightarrow 0$ as $\text{dist}(|Q|, S) \rightarrow \infty$, for the singular set of the operator. This gives control over the homogenization error for many values of Q . Another term in the error is $\|D^2u^Q\|$. In the homogeneous order one case, we have $u^Q = 0$ for $Q = 0$, so a continuity argument suggests that we may have control of $\|D^2u^Q\|$ for small values of Q . This is the case in one dimension in [10], where we obtain an analytical formula for u^Q_{xx} through (16), which gives $|u^Q_{xx}| \leq C|Q|$.

The main theorem is a formal result in the sense that it relies on the fact that u^Q is a classical solution, which does not hold in general. If F is convex (or concave), then by a famous theorem of Krylov and Evans [6, 13], or [3], $u^Q \in C^2(\mathbb{T}^d)$. However, in general we are only guaranteed $u^Q \in C^{1,\alpha}(\mathbb{T}^d)$ [12].

Proof Subtract the linearization of F at Q evaluated at $Q + D^2u^Q(y)$ from the equation for the cell problem (4), to obtain

$$\overline{F}(Q) - L^Q(Q + D^2u^Q, y) = F(Q + D^2u^Q, y) - L^Q(Q + D^2u^Q, y). \tag{11}$$

From Assumption 2.1,

$$\overline{F}(Q) - L^Q(Q + D^2u^Q, y) \leq C^+(Q, y) \frac{\|D^2u^Q\|^2}{2} \tag{12}$$

and

$$\overline{F}(Q) - L^Q(Q + D^2u^Q, y) \geq C^-(Q, y) \frac{\|D^2u^Q\|^2}{2}. \tag{13}$$

Now integrate Eqs. (12) and (13) against the invariant measure ρ^Q . This yields the upper and lower bounds (10), where we have used the fact that for all $\phi \in C^2(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} L^Q(Q + D^2\phi, y) d\rho^Q(y) = \int_{\mathbb{T}^d} F(Q, y) d\rho^Q(y), \tag{14}$$

which follows from integration by parts, since ρ^Q solves the adjoint equation (5). □

2.3 Applications of the Main Result

We give two applications of the main result. In the first example, where the operator is separable, we have an analytical formula for $\overline{L^Q}(Q)$. In this case the estimates also simplify. In the second, nonseparable example, we can find $\overline{L^Q}(Q)$ by solving a single linear homogenization problem, with coefficients given by the linearization (3).

Corollary 2.6 *Consider the separable, purely second order operator*

$$F(Q, y) = a(y)F_0(Q)$$

for $y \in \mathbb{R}^d$. Suppose that F_0 is quadratically dominated with constants $C^-(Q)$ and $C^+(Q)$. Then,

$$\overline{C}^-(Q) \leq \overline{F}(Q) - \text{HM}(a)F_0(Q) \leq \overline{C}^+(Q) \tag{15}$$

where

$$\overline{C}^\pm(Q) = \frac{1}{2}C^\pm(Q) \text{HM}(a) \int_{\mathbb{T}^d} \|D^2u^Q(y)\|^2 \frac{1}{a(y)} dy.$$

Proof 1. The formula for the linearization,

$$\overline{L^Q}(Q) = \text{HM}(a)F_0(Q)$$

follows from the Linear Homogenization Formula (Lemma 1.4).

2. From linearization, we have that $\rho^Q(y) = \text{HM}(a)/a(y)$. Using the definition, then the generalized semiconvexity/concavity constants for $F(Q, y)$ are given by

$$C^+(Q, y) = a(y)C^+(Q), \quad \text{and} \quad C^-(Q, y) = a(y)C^-(Q).$$

Passing the constants and the invariant measure into Theorem 2.4 gives the bounds provided by (15), since the coefficients $a(y)$ cancel. □

Remark 2.7 In a companion paper [10], we show that for convex operators in one dimension,

$$\overline{F}(Q) = \overline{L^Q}(Q) = \text{HM}(a)F_0(Q). \tag{16}$$

Corollary 2.8 *Consider the operator H^{a_1, a_2} given by (2). Then*

$$\begin{aligned} & \left| \overline{H^{a_1, a_2}}(Q) - \overline{L^Q}(Q) \right| \\ & \leq \frac{1}{2|\lambda_{\min}(Q) - \lambda_{\max}(Q)|} \int |a_1(y) - a_2(y)| \|D^2u^Q(y)\|^2 d\rho^Q(y) \end{aligned}$$

Proof We apply Theorem 2.4 to H^{a_1, a_2} given by (2). The linearization is given by (3). The invariant measure of the linear problem is given by the solution of (5) and the homogenized linear operator is given by (6) from Lemma 1.6.

The main step is to work out the generalized semi-concavity constants. We claim.

$$C^+(Q, x) = \frac{(a_2(x) - a_1(x))^+}{|\lambda_{\min} - \lambda_{\max}|}, \quad \text{and} \quad C^-(Q, x) = \frac{(a_1(x) - a_2(x))^-}{|\lambda_{\min} - \lambda_{\max}|}. \quad (17)$$

To prove this we proceed in steps.

1. First, take $q \in \mathbb{R}^2$ and set $f(q) = \max(q_1, q_2)$. Then $L^q(y) = \nabla f(q) \cdot y$ away from the singular set $q_1 = q_2$, since the function is homogeneous of order one. The constant $C^-(q) = 0$, since f is convex. We claim the optimal choice for $C^+(q)$ is given by

$$C^+(q) = \frac{1}{|q_2 - q_1|}$$

for $q_1 \neq q_2$. To see this, we require

$$\max(y_1, y_2) \leq \nabla f(q) \cdot y + \frac{C^+(q)}{2}|y - q|^2.$$

It is easily verified that the extremal case for the inequality occurs when $(y_1, y_2) = (q_2, q_1)$, which leads to the condition

$$|q_1 - q_2| \leq C^+(q)|q_1 - q_2|^2,$$

giving the result.

2. Let $f(q_1, q_2) = a_1 \min(q_1, q_2) + a_2 \max(q_1, q_2)$. Rewrite $f(q_1, q_2) = a_1(q_1 + q_2) + (a_2 - a_1) \max(q_1, q_2)$. We can always subtract an affine function when computing the constants. So the constants for f are the same as the constants for $(a_2 - a_1) \max(q_1, q_2)$. In this case, using the result of step 1, we obtain

$$C^+(x) = \frac{(a_2 - a_1)^+}{|q_1 - q_2|}, \quad C^-(x) = \frac{(a_1 - a_2)^-}{|q_1 - q_2|}$$

3. Next consider for the two by two matrix Q , $h(Q, x) = a_1(x) \min(q_{11}, q_{22}) + a_2(x) \max(q_{11}, q_{22})$. Then the previous step shows that the constants for h are given by the previous ones (with q_{11} replacing q_1 and q_{22} replacing q_2). Finally, since H^{a_1, a_2} depends only on the eigenvalues of Q , without loss of generality, we can choose a coordinate system where Q is diagonal when computing the generalized semiconcavity constants. It remains to show that the generalized semi-concavity condition holds for a matrix, M . If M is diagonal the condition holds. But if M is not diagonal, then the change in the norm $\|M - Q\|^2$ can be controlled by a constant, or absorbed into the definition of the norm.

□

3 Computational Setting

For our numerical experiments, we consider a wider class of separable and non-separable operators.

3.1 PDE Operators

Definition 3.1 (*Pucci-type operators*) For $\delta > 0$ and given functions $0 < \delta \leq a(y) \leq A(y)$. Write $b(y) = A(y) - a(y)$. Also write $t^+ = \max(t, 0)$. Define, for $d = 2$, the standard

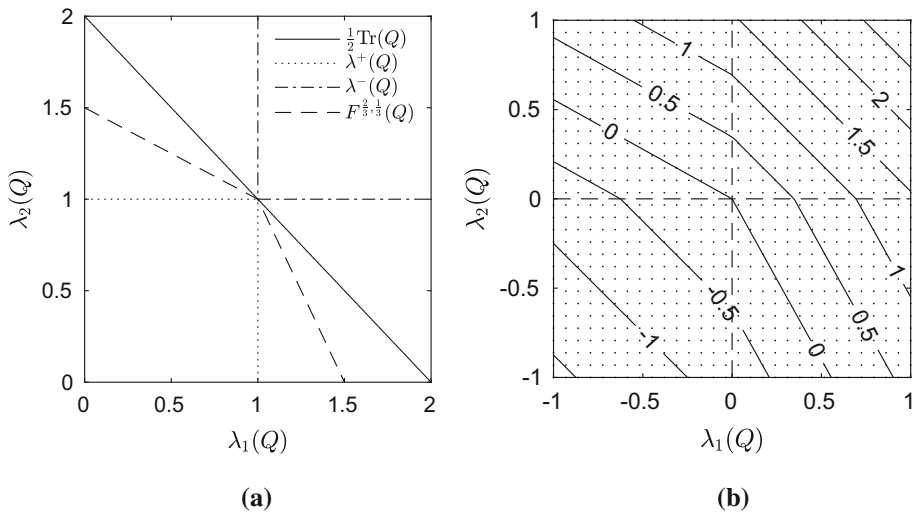


Fig. 3 **a** Level set plot of several operators as function of the eigenvalues of Q . **b** Level sets of an example Pucci operator, $P^{5/4, 2/3}(Q)$. Points indicate values of Q that were homogenized

Pucci maximal operator, the Pucci-type operator, the smoothed Pucci-type operator, and a Monge-Ampere type operator respectively as

$$P^{A,a}(Q, y) = a(y) \operatorname{Tr} Q + b(y) (\lambda_{\min}^+(Q) + \lambda_{\max}^+(Q)) \tag{18}$$

$$F^{A,a}(Q, y) = a(y) \operatorname{Tr} Q + b(y) \lambda_{\max}^+(Q). \tag{19}$$

$$F_k^{A,a}(Q, y) = a(y) \operatorname{Tr} Q + b(y) S_k(\lambda_{\min}(Q), \lambda_{\max}(Q), 0) \tag{20}$$

$$M(Q, y) = a(y) (\operatorname{Tr}(Q) + \lambda_{\min}^+(Q) \lambda_{\max}^+(Q)). \tag{21}$$

Here, for $x \in \mathbb{R}^m$, $S_k(x)$ is the smoothed maximum function,

$$S_k(x) = \frac{\sum_{i=1}^m x_i \exp(kx_i)}{\sum_{i=1}^m \exp(kx_i)}. \tag{22}$$

The function S_k goes to the max as $k \rightarrow \infty$, and to the average as $k \rightarrow 0$.

Definition 3.2 (Periodic checkerboard, stripes, and random checkerboard coefficients)

Define

$$a_0(y) = \begin{cases} 1, & y \in B \\ r, & y \in W, \end{cases} \tag{23}$$

with $r > 1$. The sets B and W are either black and white squares of a checkerboard; alternating black and white stripes of equal width; or a ‘random’ checkerboard, with black and white squares distributed with equal probability, uniformly.

Remark 3.3 (Representation and visualization of the operators) The definition above agrees with the usual definition of the Pucci operator,

$$P^{A,a}(Q, y) = \sup\{M : Q \mid a(y)I \ll M \ll A(y)I\}. \tag{24}$$

Example 3.4 For the Pucci operator $P^{A,a}(Q, x)$ given by (18), by convexity, $C^-(Q, x) = 0$ and

$$C^+(Q, x) = \begin{cases} \max \left\{ \frac{b(x)}{\text{Tr}(Q)}, \frac{A(x)}{2\lambda_{\min}} \right\}, & \text{if } \lambda_{\min}, \lambda_{\max} > 0 \\ \frac{b(x)}{2 \min(|\lambda_{\min}|, |\lambda_{\max}|)}, & \text{otherwise.} \end{cases} \tag{25}$$

3.2 Numerical Method Details

In order to compute the errors, as a function of Q , we used a grid in the $\lambda_1 - \lambda_2$ plane, and computed the linear and nonlinear homogenization at the grid points. Typical values can be seen in Fig. 3b, where black points indicate the grid values of Q tested.

We remark on the numerical methods used throughout. To compute $\bar{F}(Q)$ directly, we discretized with finite differences and solved the parabolic equation $u_t + F(Q + D^2u, y)$ using an explicit Euler method, to iteratively compute a steady state solution. We discretized using a convergent monotone scheme [15] and also using standard finite differences. The accuracy of the monotone scheme was less than the standard finite differences, so we implemented a filtered scheme [9]. In practice, the filtered scheme always selected the accurate scheme, so in this instance, perhaps because the solutions are C^2 and periodic, standard finite differences appear to converge.

For all the computations, to avoid trivial solutions, we solved with a right hand side function equal to a constant, and then subtracted the same constant from $\bar{F}(Q)$.

The computational domain was the torus $[0, 1]^2$, divided into 20×20 equal squares, each with 16 grid points per square.

Remark 3.5 (Comparison with [4]) The problem of homogenizing $a_0(y)F^{A,a}(Q)$, was considered in [4]. In their case, the spatial coefficient $a_0(y)$ varies periodically and smoothly between 2 and 3, and their homogenized value for \bar{a}_0 was 2.5 (which was the average of the coefficient $a_0(x, y) = \cos(\pi x) \cos(\pi y)$). Our results using these coefficients was $\bar{a}_0 = 2.486$, which is very close to the average. However with coefficients which are more spread out, we obtain values far from the average.

4 Numerical Results

4.1 Numerical Results: Separable Operators

Here we check the homogenization error of the bound for separable operators in two dimensions, from Corollary 2.6. We are in the convex case, so the lower bound is zero.

We performed numerical simulations on four operators, see Definition 3.1.

- $a_0(y)P^{3,1}(Q)$
- $a_0(y)F^{3,1}(Q)$
- $a_0(y)F_k^{3,1}(Q)$, with $k = 10$ and $k = 0.1$
- $a_0(y)M(Q)$

In Fig. 4 we compare the error $\bar{F}(Q) - \bar{L}^Q(Q)$ for a separable Pucci operator on a checkerboard, we also illustrate the constant $C^+(Q, y)$. In this case we have an analytical formula for $\bar{L}^Q(Q)$. This figure illustrates the Main Theorem: when the constant is large the error from the linearization is high. The error is less than $1e-6$ outside of a small region about the axis, and on the order of 0.1 near the axis.

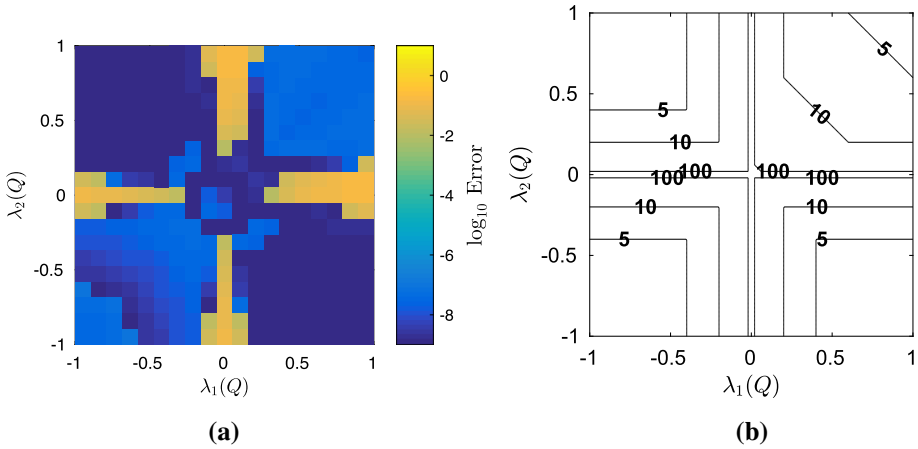


Fig. 4 Homogenization of a separable Pucci example operator, $a(y)P^{3,1}$, on a periodic checker board, with coefficients of 1 or 2 ($r = 2$). **a** Error $\bar{F}(Q) - \bar{L}^Q(Q)$. The error is $1e-6$ or less in the blue part of the domain. In the yellow region it goes from 0.01 up to 0.15. The regions where the error is small coincide with smaller values of the semi-concavity constant. **b** An upper bound of the semi-concavity constant $C^+(Q, y)$ (Color figure online)

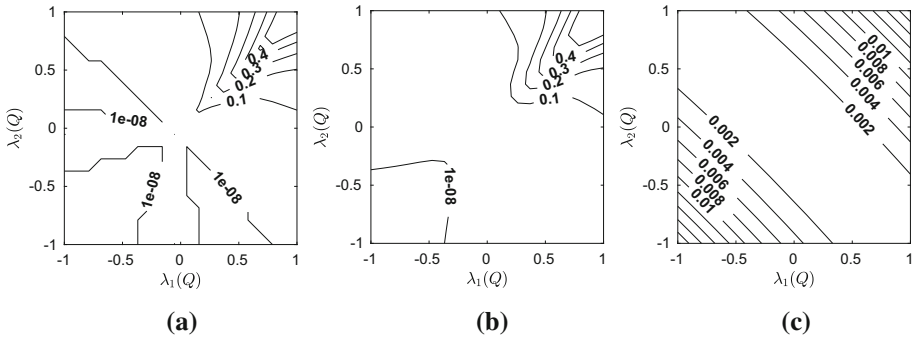


Fig. 5 Homogenization error for a smoothed Pucci type operator. The coefficients $a(y)$ are on a checker board with $r = 2$ (i.e. $a = 1$ or 2). The operators are defined in Sect. 4. **a** error on a Pucci like operator. **b**, **c** error on a smoothed Pucci like operator. **a** $a_0(y)F^{3,1}(Q)$, $r = 2$, **b** $a_0(y)F_{10}^{3,1}(Q)$, $r = 2$, **c** $a_0(y)F_{0.1}^{3,1}(Q)$, $r = 2$

In Fig. 5, we show how the error $\bar{F}(Q) - \bar{L}^Q(Q)$ decreases as the operator becomes smoother. The operator with the smallest maximum curvature (Fig. 5c) exhibits the smallest error. As the operator becomes less smooth, the error increases. For the two sharper operators, there is still very high accuracy away from the highest curvature regions. We see that error of the smooth operator, $F_{10}^{3,1}(Q)$, is slightly smaller than the non smooth operator’s error near the line $\lambda_1 = \lambda_2$ (this is where the non smooth operator is not differentiable). As the smoothing constant $k \rightarrow 0$, the error of the linearized homogenization decreases. For example, setting $k = 0.1$, as in Fig. 5c, results in an error on the order of 0.01. In all cases, the error is near zero in a large part of the domain. It concentrates near the positive diagonal, where it is .1–.4 for the non-smooth operator, and similar for the operator with a small smoothing parameter. A larger smoothing parameter sends the error in a similar region to the range .002–0.01. A small amount of smoothing has a small effect on the error. More smoothing leads to errors going from .1 to .002 in a similar part of the domain.

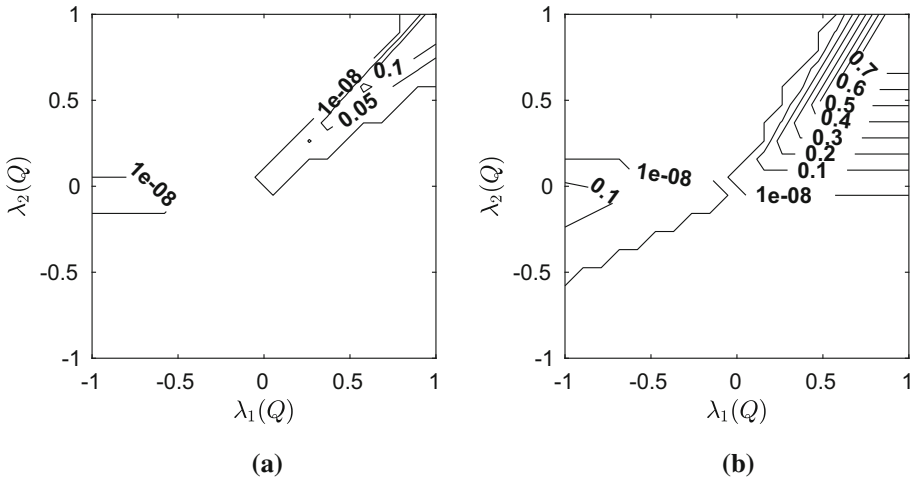


Fig. 6 Error for $a_0(y)F^{3,1}$ on stripes, with different ratios r . **a** $r = 1.2$, **b** $r = 2$

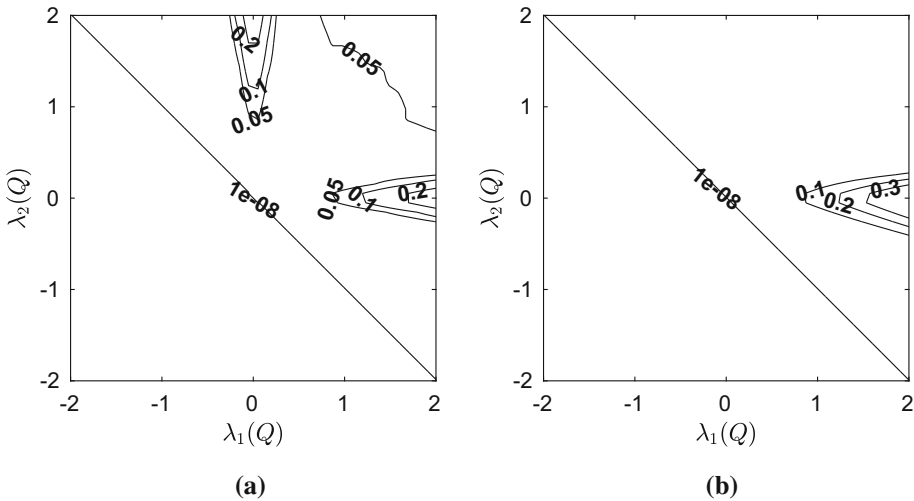


Fig. 7 Error for $M(Q, y)$ with $r = 2$, on a periodic checkerboard and on stripes. **a** Checkerboard, **b** stripes

Figure 6 presents the error for $a_0(y)F^{3,1}(Q)$ on stripes. On stripes, the regions with large error are much smaller than the operators on checkerboard. We hypothesize that this is because stripes have a smoothing effect. The location where the large error is located depends on the interplay between the operator and the direction of the stripes. Given that in this example the homogenized operator is $\mathcal{O}(1)$, the error here is particularly large. In a companion paper, we will derive a closer lower bound for $F^{A,a}(Q)$, using the optimal invariant measure of the nonlinear operator.

Figure 7 shows error for $a_0(y)(\text{Tr}(Q) + \text{MA}(Q))$ on both stripes and a periodic checkerboard. For the Monge Ampere type operator on checkerboard, error is on the order of $1e-2$ in the first quadrant, where the curvature is bounded. Elsewhere the error is negligible.

As r (the scaling coefficient of $a_0(y)$) grows, so does the error. As expected, for the two Pucci type operators on checkerboard, away from the regions where the curvature is

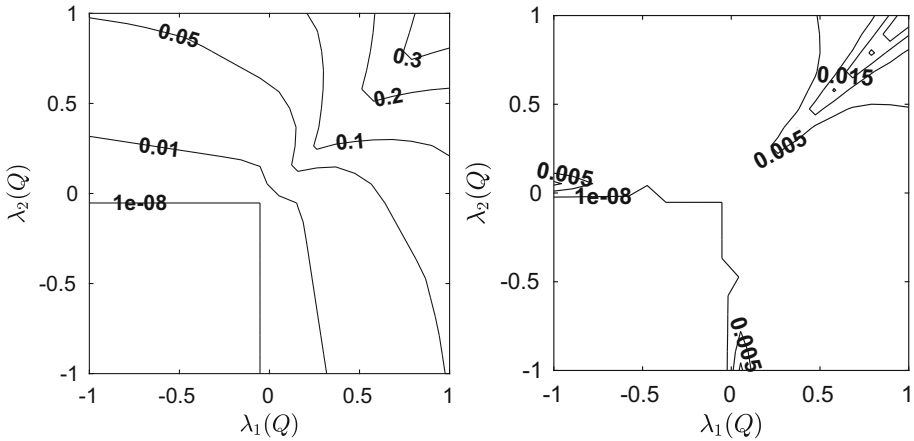


Fig. 8 Error for the non-separable operator on a periodic checkerboard. Left: alternating between $F^{1,1}$ and $F^{4,1}$. Right: alternating between $F^{2,1}$ and $F^{1,3/4}$

unbounded, the error is negligible: this is where the operators are linear. Although we do not show it, for all figures, the error profile on the random checkerboard is nearly identical to the periodic checkerboard.

4.2 Numerical Results: Non-separable Operators

Now we consider nonseparable coefficients for $F^{A,a}(Q, y)$, refer to Definition 3.1.

For both periodic and random checkerboard coefficients, the numerically computed values of $\overline{F^{A,a}(Q)}$ depend only on the eigenvalues of Q , not on the eigenvectors. In addition, $\overline{F^{A,a}(Q)}$ is homogeneous order one. So the entire function $\overline{F^{A,a}(Q)}$ is determined by the 1-level set of $\overline{F^{A,a}(Q)}$ for diagonal matrices Q .

We write

$$\overline{L^Q}(Q) = \overline{A}\lambda^+(Q) + \overline{a}\lambda^-(Q) \tag{26}$$

where the coefficients are obtained by numerical homogenization of the linearized operator (3) when Q had at least one positive eigenvalue. (In the negative definite case the operator is linear and the error was insignificant).

We found that error was within 5% for a range of values of A and a with coefficients which vary by a factor of 10.

In Fig. 8 we show the error on a periodic checkerboard, with

$$F^{A,a}(Q, y) = \begin{cases} \text{Tr}(Q), & y \in B \\ F^{4,1}, & y \in W. \end{cases}$$

The error is on the order of $1e-1$ near the line $\lambda^+ = \lambda^-$ in the first quadrant; on the order of $1e-2$ in the second and fourth quadrants; and negligible otherwise. In Fig. 8 we plot the error against the numerically homogenized value for a nonconvex operator alternating between $F^{2,1}$ and $F^{1,3/4}$ on a periodic checkerboard.

4.2.1 Further Experiments

We let A and a each take two positive values in periodic checkerboard pattern. In the second, we let A and a each take two positive values in a random checkerboard, drawn randomly from a Bernoulli trial with probability p . We checked both when $p = \frac{1}{2}$ and other values of p . When $p = \frac{1}{2}$ the homogenized operator on the random checkerboard is identical to the homogenization on the periodic checkerboard. Finally, we also checked the case when A and a are each drawn from a uniform distribution with positive support. In all of these cases, the numerically homogenized operator is (numerically) isotropic, homogeneous order one, and agrees closely with \overline{F} in the approximate formula (26).

5 Conclusions

We studied the error between the homogenization of the linearized operator and the fully nonlinear homogenization. We obtained upper and lower bounds on the error in terms of the generalized semiconcavity constants of the operator.

We also performed numerical calculations. For the class of operators we studied, linearization was very accurate for a wide range of values of Q , with negligible error in some cases. The numerically computed errors were small, and concentrated around regions of high curvature in Q of the operator $F(Q, x)$. Errors grew with the degree of nonlinearity and with the range of the coefficients.

The numerical results are consistent with the bounds, although in some cases the error was smaller than was predicted by the bounds.

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